# Analytical Study of Heat Transfer Rates for Parallel Flow of Liquid Metals Through Tube Bundles: II

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**A theoretical analysis of heat transfer to liquid metals in parallel flow through a tube bundle, following the method of Lyon, results in an equation which agrees within 10% with the calculated values. The conditions are fully developed turbulent flow, constant heat flux at the wall, and an infinite number of tubes arranged on an equilateral triangular pitch. The hexagonal flow area assignable to any tube is approximated by a circle of equal area, and the velocity distribution of Bailey for an annulus is assumed to apply from the inner wall to the circle of maximum flow. The range of conditions is: pitch-to-diameter ratio, 1.375 to 10; Reynolds number, $10^4$ to $10^6$; Prandtl number, 0 to 0.1; and Peclet number, 0 to $10^5$. Laminar and slug flow Nusselt numbers are also determined.**

Heat transfer correlations for ordinary fluids do not apply to liquid metals. With ordinary fluids the molecular conductivity of heat is negligible compared with the eddy conductivity as a means of transporting heat in the turbulent core. However with liquid metals this is no longer true, owing to their extremely large molecular conductivities.

Martinelli (*1*) was the first to take this factor into account in developing a heat transfer relation for low Prandtl number fluids flowing within tubes. Lyon (*2*) made similar assumptions, but followed a different development, and presented his results in the form of a simpler equation. Other investigators have assumed constant wall temperature instead of uniform heat flux and have developed correlations for liquid metal heat transfer in flow between parallel plates and in annuli, as well as in tubes (*3*). In addition, a number of experimental liquid metal heat transfer studies have been made, as reviewed by Lubarsky and Kaufman (*4*).

In the present paper a method similar to that employed by Lyon has been used for an analytical investigation of the heat transfer to a liquid metal in parallel flow through a bundle of tubes. Constant heat flux was employed, and the outer boundary was approximated by a circle of equal area. A simultaneous investigation of this problem was made independently by Dwyer and Tu (*5*) but with different variables and different values of the parameters and with the integrations performed at different stages in the development. A further and more significant difference in the two developments was in the assumption of the velocity distribution. In both studies the model of an annulus was used taking the flow from the inner wall to the point of maximum flow. However Dwyer and Tu used the recent correlations of Rothfus, Walker, and Whan (*6*), whereas in the present paper the velocity distribution assumed by Bailey (*7*) was used. The results of the present investigation show Nusselt numbers from 7 to 30% higher than those obtained by Dwyer and Tu in the pitch-to-diameter ratio range of 1.375 to 2.2.

## METHOD

It is postulated that the physical properties of the fluid are constant, that the system is operating under steady state conditions with no end effects, and that there is uniform radial heat flux at the tube wall. This means that in the region being considered the velocity and temperature profiles are fully established so that $\partial t/\partial x$ is constant at all points.

Three of the tubes in an infinite array with uniform triangular pitch are shown in Figure 1. The hexagon circumscribed about tube (a) represents the flow cross section and fluid assignable to that particular tube. It is assumed that the actual situation can be closely enough approximated by replacing the hexagon with a circle of equal area (thus maintaining the same hydraulic radius) with the velocity and temperature profiles that would be obtained in the case of the shear stresses and heat flux dropping to zero at the circumference of the circle.

By definition the rate of heat flow from a unit length of tube is given by

$$q_1 = h(t_1 - t_m)2\pi r_1 \tag{1}$$

By a heat balance

$$q_1 = \pi(r_m^2 - r_1^2)u_a \rho c_p(\partial t_m/\partial x) \tag{2}$$

The bulk temperature is defined by

$$t_m = \frac{\int_{r_1}^{r_m} rutdr}{\int_{r_1}^{r_m} rudr} \tag{3}$$

Hence

$$(t_1 - t)_m = \frac{2\int_{r_1}^{r_m} (t_1 - t)rudr}{u_a(r_m^2 - r_1^2)} \tag{4}$$

At any radius

$$t_1 - t = -\int_{r_1}^{r} (\partial t/\partial r)dr \tag{5}$$

By definition

$$q = -k_{eff.} 2\pi r(\partial t/\partial r) \tag{6}$$

A heat balance around a cylindrical shell of inner radius $r$ gives

$$q = 2\pi \rho c_p(\partial t/\partial x) \int_r^{r_m} rudr \tag{7}$$

Combining Equations (7) and (6) and substituting the result in (5), then substituting that result in (4), and finally substituting that result and (2) in (1), one gets

$$h = \frac{(r_m^2 - r_1^2)^2 u_a^2}{4r_1 \int_{r_1}^{r_m} ru\left\{\int_{r_1}^{r}\left[\frac{\int_r^{r_m} rudr}{rk_{eff.}}\right]dr\right\}dr} \tag{8}$$

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The effective thermal conductivity is given by

$$k_{eff.} = k + k_e \quad (9)$$

$$(k_{eff.}/k) = [(k/c_p\rho) + \epsilon_H](c_p\rho/k) \quad (10)$$

The terms in brackets are the molecular and eddy diffusivities of heat. Dividing and multiplying by the molecular diffusivity of momentum (kinematic viscosity) one obtains

$$(k_{eff.}/k) = [(1/N_{Pr}) + (\epsilon_H/\nu)]N_{Pr} = 1 + \psi N_{Pr}(\epsilon_M/\nu) \quad (11)$$

The equivalent diameter is defined by

$$D_e = [4\pi(r_m^2 - r_1^2)/2\pi r_1] \quad (12)$$

Equation (8) may be converted to a dimensionless form by substituting the Nusselt number based on equivalent diameter, Equation (10), and dimensionless velocity and radius ratios

$$\frac{1}{N_{Nu}} = \frac{2}{(R_m^2-1)^3}\int_1^{R_m} RV \left[\int_1^R \frac{\int_R^{R_m} RVdR}{R\left(1+\psi N_{Pr}\frac{\epsilon_M}{\nu}\right)}dR\right]dR \quad (13)$$

The number of integrations may be reduced from three to two. When one introduces subscripts to designate the variable in each integration, Equation (13) may be written as

$$\frac{1}{N_{Nu}} = \frac{2}{(R_m^2-1)^3}\int_1^{R_m}\int_1^{R_t}\int_{R_q}^{R_m} \frac{R_t V_t R V}{R_q\left(1+\psi N_{Pr}\frac{\epsilon_M}{\nu}\right)} dRdR_q dR_t \quad (14)$$

By changing the order of the integrations with respect to $R_q$ and $R_t$, with corresponding changes in the limits

$$\frac{1}{N_{Nu}} = \frac{2}{(R_m^2-1)^3}\int_1^{R_m}\int_{R_q}^{R_m}\int_{R_q}^{R_m} \frac{R_t V_t R V}{R_q\left(1+\psi N_{Pr}\frac{\epsilon_M}{\nu}\right)} dRdR_t dR_q \quad (15)$$

When one combines the identical integrals

$$\frac{1}{N_{Nu}} = \frac{2}{(R_m^2-1)^3}\int_1^{R_m} \frac{\left(\int_{R_q}^{R_m} RVdR\right)^2}{R_q\left(1+\psi N_{Pr}\frac{\epsilon_M}{\nu}\right)} dR_q \quad (16)$$

Equation (16) is a general expression by which the heat transfer coefficient may be calculated with the knowledge of the distribution of $V$ and $\psi$ with respect to $R$. The velocity profile for turbulent flow may be estimated from shear profile, in accordance with the method proposed by Bailey (*7*) for the velocity profile in annuli between the inner wall and radius of maximum velocity.

The eddy diffusivities of heat and momentum are usually assumed equal in theoretical analyses of turbulent heat transfer, on the basis of the analogy between heat and momentum transfer. In terms of the Prandtl mixing-length concept a particle of fluid is assumed to travel the same distance normal to the direction of flow before losing its identity and before attaining a different temperature. Two evaluations of $\psi$ have been reported based on experimental velocity and temperature profiles for mercury in tubes, and the results differ considerably. Isakoff and Drew (*8*) found $\psi$ to vary radially, the average value ranging from about 0.5 at $N_{Re}$ = 40,000 to about 1.5 at $N_{Re}$ = 400,000, whereas Brown et al. (*9*) found little radial variation except in the vicinity of the wall, with $\psi$ ranging from about 0.65 at $N_{Re}$ = 250,000 to about 0.85 at $N_{Re}$ = 730,000. The Nusselt numbers from both investigations are in fair agreement with those predicted by Martinelli (*1*) and Lyon (*2*) with $\psi = 1.0$ assumed, and so $\psi$ will be assumed constant at 1.0 in this report.

Consider a force balance on a cylindrical shell of fluid of unit length, with outer radius $R_m$ and inner radius $r$:

$$\tau = -\frac{r_m^2 - r^2}{2r}\frac{dp}{dx} \quad (17)$$

$$\frac{\tau}{\tau_1} = \frac{(r_m^2 - r^2)r_1}{(r_m^2 - r_1^2)r} = \frac{R_m^2 - R^2}{(R_m^2-1)R} \quad (18)$$

The shear stress is not a linear function of the distance from the wall as it is in tubes, and so the universal velocity-distribution relations for flow within tubes must be modified. The friction factor is defined by

$$\tau_1 = \frac{f\rho u_a^2}{2g_c} \quad (19)$$

and the friction velocity by

$$u^* = \sqrt{(g_c\tau_1)/\rho} = u_a\sqrt{f/2} \quad (20)$$

The friction-factor relationship for flow within tubes is generally considered to be applicable to ducts of any cross-sectional shape if the diameter of the tube is replaced by the equivalent diameter of the duct (*10*). Experimental data for parallel flow through rod bundles show considerable scatter, with some experimenters finding good agreement with the results for flow within tubes (*11, 12, 13*) and others finding friction factors 65% (*14*) to 100% (*15*) higher. The standard friction-factor relationship for flow within tubes will be used in this report. This assumption should be best at ordinarily low values of the pitch-to-diameter ratio, as the friction factor might be expected to become higher as the pitch-to-diameter ratio is increased. The velocity profile in the turbulent zone for flow within tubes is given by

$$u^+ = C' + 2.5 \ln y^+ \quad (21)$$

where

$$y^+ = (y/r_1)(N_{Re}/2)(u^*/u_a) \quad (22)$$

and

$$u^+ = u/u^* \quad (23)$$

For flow within tubes $C'$ = 5.5. The term $(y/r_1)$ in Equation (22) may be replaced by the equivalent shear distribution to give

$$y^+ = \left(1 - \frac{\tau}{\tau_1}\right)(N_{Re}/2)(u^*/u_a) \quad (24)$$

It is now assumed that Equation (24) which is for inside tubes is the proper definition of $y^+$ to use with Equation (21) to determine the velocity distribution in the turbulent zone for a fluid in parallel flow through a tube bank. Bailey (*7*) found that a similar approach checked fairly well with Knudsen and Katz's (*16*) experimental data for flow in an annulus of diameter ratio 3.60.

It was also assumed by Bailey that the velocity distribution in the laminar and transition zones is given by substituting Equation (24) into the equations for flow in the laminar and transition zones of a tube. This assumption is incorrect, though it makes little difference in the final results. The actual velocity distribution in the laminar zone is found by integrating the viscous flow equation

$$du/dr = g_c\tau/\rho\nu \quad (25)$$

from the wall to any $R$, and substituting Equations (18), (19) and the defining equation for the Reynolds number

$$N_{Re} = \frac{2(r_m^2 - r_1^2)u_a}{r_1\nu} \quad (26)$$

The resulting flow equation for the laminar zone is

$$V = \frac{fN_{Re}(2R_m^2 \ln R - R^2 + 1)}{8(R_m^2-1)^2} \quad (27)$$

The transition zone will be neglected in these calculations, as it has little effect on liquid metal heat transfer. It is assumed that the transition between laminar and turbulent flow occurs at the point at which the calculated velocities for the two zones match.

The ratio of eddy to molecular diffusivity of momentum, $\epsilon_M/\nu$, in the turbulent core is found by differentiating Equation (21) with respect to $r$ and substituting the values of $u^+$ and $y^+$ from Equations (23) and (24) and $u^*$ from Equation (20):

$$du/dr = -\frac{2.5\,u_a\sqrt{f/2}}{\tau_1 - \tau}\frac{d\tau}{dr} \qquad (28)$$

From Equation (18)

$$\frac{d\tau}{dr} = -\frac{\tau_1 r_1 (r^2 + r_m^2)}{r^2(r_m^2 - r_1^2)} \qquad (29)$$

The eddy diffusivity of momentum is defined by

$$\frac{du}{dr} = \frac{g_c \tau}{\rho(\epsilon_M + \nu)} \qquad (30)$$

Combining Equations (28), (29), and (30) and substituting Equations (18) and (20) one gets

$$\epsilon_M + \nu = \frac{u_a\sqrt{f/2}\,r(r_m^2 - r^2)\left(1 - \frac{\tau}{\tau_1}\right)}{2.5(r_m^2 + r^2)} \qquad (31)$$

Dividing by $\nu$ and substituting Equation (26) one obtains

$$\frac{\epsilon_M + \nu}{\nu} = \frac{r_1 r(r_m^2 - r^2)}{2.5(r_m^2 - r_1^2)(r_m^2 + r^2)}\left[\frac{\left(1 - \frac{\tau}{\tau_1}\right)N_{Re}\sqrt{f/2}}{2}\right] \qquad (32)$$

The term in brackets is the authors' definition of $y^+$, and in terms of the corresponding dimensionless quantities Equation (32) becomes

$$\frac{\epsilon_M + \nu}{\nu} = \frac{R(R_m^2 - R^2)y^+}{2.5(R_m^2 - 1)(R_m^2 + R^2)} \qquad (33)$$

Equation (33) applies to the turbulent zone. For the laminar zone

$$\frac{\epsilon_M}{\nu} = 0 \qquad (34)$$

The internal integral in Equation (16), $\int_{R_q}^{R_m} RVdR$, may be evaluated analytically for various values of $R_q$. Substituting Equations (23), (24), and (18) in (21) one obtains

$$V = C + 2.5\sqrt{f/2}\ln\frac{N_{Re}}{2}\sqrt{f/2}\;\frac{(R-1)(R+R_m^2)}{(R_m^2 - 1)R} \qquad (35)$$

where $C = C'\sqrt{f/2}$. Multiplying by $R$ and integrating one gets

$$\int_{R_q}^{R_m} RVdR = \frac{C}{2}(R_m^2 - R_q^2) + 1.25\sqrt{f/2}[(R_m^2 - R_q^2)\ln\frac{N_{Re}\sqrt{f/2}}{2(R_m^2 - 1)} + (R_m^2 - 1)\ln(R_m - 1) - (R_q^2 - 1)\ln(R_q - 1) - \frac{R_m^2 - R_q^2}{2} + (R_m^2 - 1)(R_m - R_q) - (R_m^4 - R_m^2)\ln(R_m + R_m^2) + (R_m^4 - R_q^2)\ln(R_q + R_m^2) - R_m^2\ln R_m + R_q^2\ln R_q] \qquad (37)$$

Of the terms within the brackets in Equation (37) the first is the only one which varies with the flow rate.

To evaluate Equation (16) values of $P/D$, $N_{Re}$ and $\psi N_{Pr}$ are assumed. The friction factor is obtained from the relation for flow within tubes:

$$f = 0.00140 + \frac{0.125}{N_{Re}^{.32}} \qquad (38)$$

$R_m$ is calculated from

$$R_m^2 = \frac{2\sqrt{3}}{\pi}(P/D)^2 \qquad (39)$$

Equation (18) may be solved for $R$

$$R = -(\tau/\tau_1)\frac{(R_m^2 - 1)}{2} + \sqrt{(\tau/\tau_1)^2\frac{(R_m^2 - 1)^2}{4} + R_m^2} \qquad (40)$$

and $R$ evaluated for values of $\tau/\tau_1$ between 0 and 1.

The constant $C$, which is necessary for the evaluation of Equation (37), is found by trial and error as follows: Unadjusted values of the integral $\int_{R_q}^{R_m} RV_u dR$ are calculated from Equation (37), where $V_u$ is defined by

$$V_u = V - C \qquad (41)$$

in the turbulent zone. A value for $C$ is assumed, and the transition point between the laminar and turbulent zones, $R_c$, is determined as the larger value of $R$ for which Equations (27) and (35) give the same value for $V$. $C$ is now adjusted to satisfy the identity

$$\frac{\int_1^{R_m} RVdR}{\int_1^{R_m} RdR} = 1 \qquad (42)$$

The laminar and turbulent regions are evaluated separately

$$\int_1^{R_m} RVdR = \int_1^{R_c} RVdR + \int_{R_c}^{R_m} RVdR = \int_1^{R_c} RVdR + \int_{R_c}^{R_m} RV_u dR + C\int_{R_c}^{R_m} RdR \qquad (43)$$

where the first integral on the right is found from Equation (27)

$$\int_1^{R_c} RVdR = \frac{f N_{Re}}{32(R_m^2 - 1)^2}[2R_m^2 R_c^2(2\ln R_c - 1) + 2R_m^2 - R_c^4 + 2R_c^2 - 1] \qquad (44)$$

and the second integral from Equation (37). When one solves Equations (42) and (43) for $C$

$$C = \frac{2}{R_m^2 - R_c^2}\left(\frac{R_m^2 - 1}{2} - \int_1^{R_c} RVdR - \int_{R_c}^{R} RV_u dR\right) \qquad (45)$$

If this value of $C$ differs from that initially assumed, a new value of $R_c$ is calculated and the process repeated. The adjusted values of the integral $\int_{R_q}^{R_m} RVdR$ are found by adding $(C/2)(R_m^2 - R^2)$ to the unadjusted integral $\int_{R_q}^{R_m} RV_u dR$. Values of $\epsilon_M/\nu$ are calculated from Equation (33), and the term within the large integral of Equation (16) is evaluated for various values of $R$. This integral is determined graphically, and the Nusselt number is calculated from Equation (16).

## RESULTS

Nusselt numbers were calculated, with $\psi = 1$, for $P/D$ ratios of 1.375, 2, 3, and 10; Reynolds numbers of $10^4$, $10^5$, and $10^6$; and Prandtl numbers of 0, 0.001, 0.01, and 0.1. At $N_{Re} = 10^5$ for $P/D = 10$ and at $N_{Re} = 10^4$ for $P/D = 3$ the laminar zone was found to extend over most of the flow region, indicating that the flow would be laminar or transitional, and so no heat transfer calculations were made for these conditions. A total of thirty-six Nusselt numbers were calculated and are presented in Figure 2.

As is usual with liquid metal heat transfer the Nusselt number correlated quite well with the Peclet number for any geometry. This is due to the high thermal conductivity which adds to the turbulence as a means of transporting heat and reduces the relative importance of the laminar film at the wall as a barrier to heat flow.

Lyon (2) proposed the following equation, based on calculations similar to those in this report, to correlate liquid metal heat transfer for flow within tubes:

$$N_{Nu} = 7.0 + 0.025(\psi N_{Pe})^{0.8} \qquad (46)$$

Equations of similar form have been developed for flow in annuli and between parallel plates. The first term on the right-hand side of Equation (46) represents the molecular conduction contribution to the heat transfer, while the second term represents the contribution of eddy conduction. As may be seen by examining the Nusselt numbers for the case in which the eddy conductivity is neglected ($N_{Pr} = 0 = N_{Pe}$), the molecular conductivity contribution is not quite constant but increases by about 6.5% for a Reynolds number change from $10^4$ to $10^5$ and increases an additional 1.1% for a Reynolds number increase to $10^6$, owing to the change in velocity profile. However it was found that all of the data could be represented by the following empirical equation:

$$N_{Nu} = 7.0 + 3.8(P/D)^{1.52} + 0.027(P/D)^{0.27} (\psi N_{Pe})^{0.8} \quad (47)$$

This equation is plotted in Figure 2 together with the calculated Nusselt numbers. The agreement is within 10% for all the points and within 5% for all but four points.

In an independent investigation of this problem, for pitch-to-diameter ratios of 1.375, 1.7, and 2.2, and values of $\psi N_{Pe}$ ranging from 60 to 10,000, Dwyer and Tu (5) obtained the following equation:

$$N_{Nu} = 0.93 + 10.81(P/D) - 2.01(P/D)^2 + 0.0252(P/D)^{0.273} (\psi N_{Pe})^{0.8} \quad (48)$$

Equations (47) and (48) are compared in Figure 3 for a pitch-to-diameter ratio of 1.375. The Hartnett and Irvine (17) approximation, described later, is also shown. The predicted Nusselt numbers of the present report are about 9% higher than those of Dwyer and Tu at a pitch-to-diameter ratio of 1.375 and about 25% higher at a pitch-to-diameter ratio of 2.2. The Hartnett and Irvine approximation lies between the other two equations over a substantial part of the usual working range of Peclet numbers and for this range of pitch-to-diameter ratios.

The calculation procedure used in the present work is believed to lead to greater precision than that achieved by Dwyer and Tu, since in the present work there are fewer graphical integrations and differentiations and more use of dimensionless groups to reduce the number of apparent variables than in Dwyer and Tu's report. However the effect of calculation errors in Dwyer and Tu's report appears to be slight, the Nusselt number calculations being reproducible to about 2½%. The major source of the discrepancy between the present results and those of Dwyer and Tu appears to be the different velocity profiles which were used in the calculations. In Figure 4 sample calculated velocity profiles used in the two reports are compared. The correlation of Rothfus et al (6), which was used by Dwyer and Tu, was obtained from measurements of a total of seven velocity profiles in three annuli, and Knudsen and Katz's (16) data, upon which the velocity profile correlation used in the present study was based, were taken from six velocity profile measurements on one annulus.

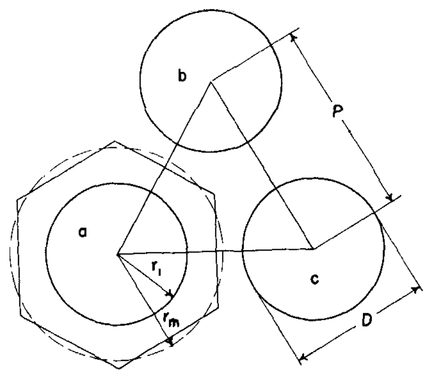


Fig. 1. Cross section of three-tube portion of tube bundle showing spatial arrangement of tubes.

## SQUARE ARRAY

In the previous section the hexagonal flow area associated with a given tube was approximated by a circle of equal area. Similarly the square flow area associated with a given tube in a square array of tubes may be approximated by a circle of equal area, though this leads to greater error than in the case of equilateral triangular spacing.

For a square array of tubes the following relation holds:

$$R_m^2 = \frac{4}{\pi} (P'/D)^2 \quad (49)$$

Combining Equations (49) and (39) one gets

$$(P/D)^2 = \frac{2}{\sqrt{3}} (P'/D)^2 \quad (50)$$

Equation (47) holds for both equilateral triangular and square pitch arrays for a given value of $R_m$. Therefore substitution of Equation (50) in (47) gives the equivalent heat transfer equation for a square array of tubes:

$$N_{Nu} = 7.0 + 4.24(P'/D)^{1.52} + 0.0275(P'/D)^{0.27} (\psi N_{Pe})^{0.8} \quad (51)$$

Unpublished calculations, in which slug flow Nusselt numbers for the actual geometry and for the circular approximation boundary are compared, show agreement within 5% for pitch-to-diameter ratios above 1.3 with equilateral triangular tube spacing, and for pitch-to-diameter ratios above 1.7 with square spacing.

## SLUG FLOW

By assuming slug flow ($V = 1.0$, $\epsilon = 0$) Equation (16) may be integrated to obtain

$$N_{Nu,\,s} = \frac{8(R_m^2 - 1)^3}{4R_m^4 \ln R_m - 3R_m^4 + 4R_m^2 - 1} \quad (52)$$

Equation (52) has been evaluated for a number of values of $R_m$, and Figure 5 shows the results.

Slug flow Nusselt numbers are close to but somewhat higher than the turbulent flow Nusselt numbers with the eddy conductivity term omitted, since the turbulent flow velocity profiles are fairly flat but drop to zero at the walls. Hartnett and Irvine (17) have proposed that in the absence of a theoretical analysis for liquid metal heat transfer in a duct of any given geometry the heat transfer equation be estimated as

$$N_{Nu} = \frac{7}{8} N_{Nu,\,s} + 0.025\, N_{Pe}^{0.8} \quad (53)$$

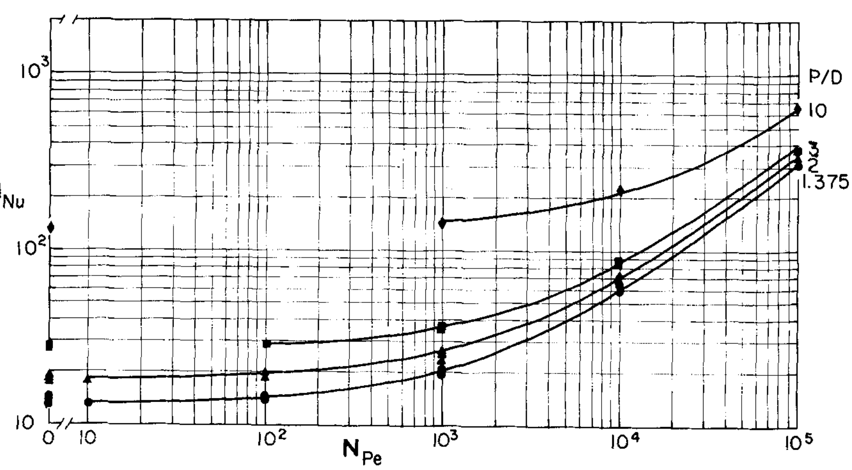


Fig. 2. Calculated Nusselt numbers and curves from Equation (47).

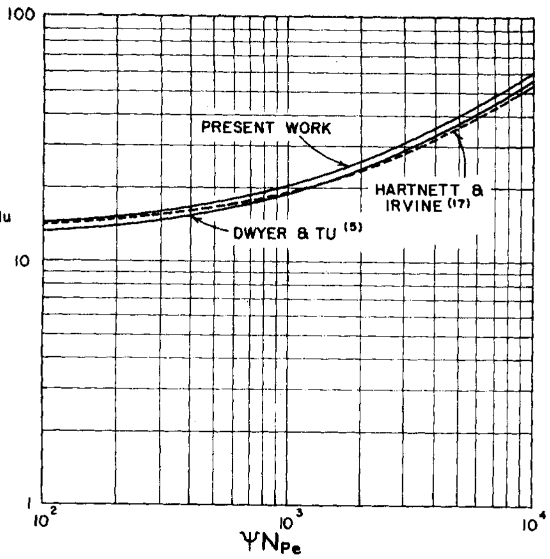


**Fig. 3. Comparison of predicted Nusselt numbers from present work, Dwyer and Tu (5), and Hartnett and Irvine (17), for pitch-to-diameter ratio 1.375.**

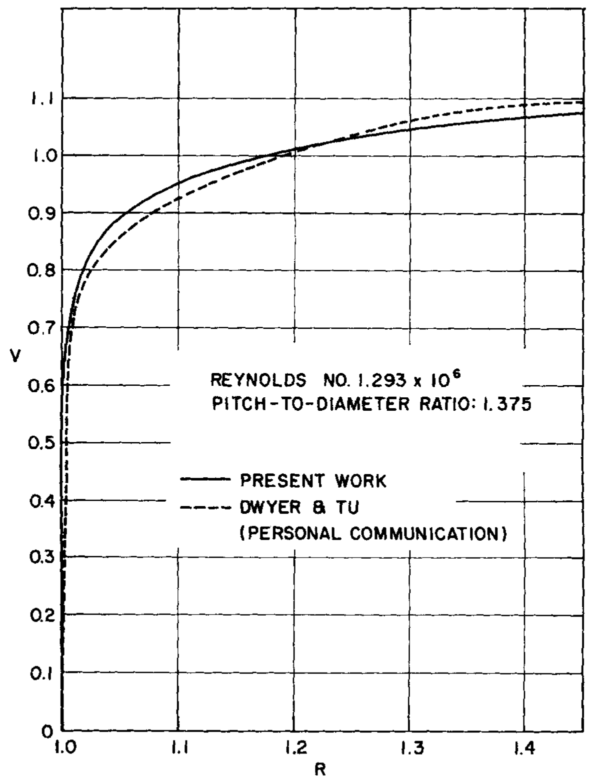


**Fig. 4. Comparison of sample velocity distributions assumed in present work with those assumed by Dwyer and Tu (personal communication.)**

Equation (53) is compared with Equations (47) and (48) for $P/D = 1.375$ in Figure 3. This equation is in fair agreement with the results of the present analysis at low values of $P/D$ but breaks down at higher values owing to the absence of a geometrical correction factor on the term containing the Peclet number.

## LAMINAR FLOW

For liquid metals the heat transfer coefficients for viscous flow are an appreciable fraction of the coefficients obtained at the lower turbulent flows.

For laminar flow Equation (27) applies over the entire flow region. Equation (44) applies throughout, with the upper limit of the integration becoming $R_m$ and each $R_e$ being replaced by $R_m$. Substituting Equation (44) in (42) one gets

$$fN_{Re} = \frac{16(R_m^2 - 1)^3}{4R_m^4 \ln R_m - 3R_m^4 + 4R_m^2 - 1} \quad (54)$$

When one substitutes Equations (54) and (27) in (16) with $\epsilon_M = 0$, the laminar flow Nusselt number may be determined by performing the integrations in Equation (16):

$$N_{Nu,\,L} = [144(R_m^2 - 1)(4R_m^4 \ln R_m - 3R_m^4 + 4R_m^2 - 1)^2]/[R_m^8(1{,}152 \ln^3 R_m - 2{,}592 \ln^2 R_m + 2{,}280 \ln R_m - 719) + R_m^6(1{,}152 \ln^2 R_m - 2{,}880 \ln R_m + 1{,}680) + R_m^4(720 \ln R_m - 1{,}296) + 368 R_m^2 - 33] \quad (55)$$

Equation (55) has been evaluated for various values of $R_m$, and the results are shown in Figure 5.

## ACKNOWLEDGMENT

The assistance of Mr. Govindasharma Srikantiah in performing many of the calculations is gratefully acknowledged.

## NOTATION

$C$ = $C'\sqrt{f/2}$ = constant in velocity equation, dimensionless
$C'$ = constant in velocity equation, dimensionless
$c_p$ = specific heat, B.t.u./(lb.-mass) (°F.)
$D$ = diameter of tubes, ft.
$D_e$ = $4m$ = equivalent diameter, ft.

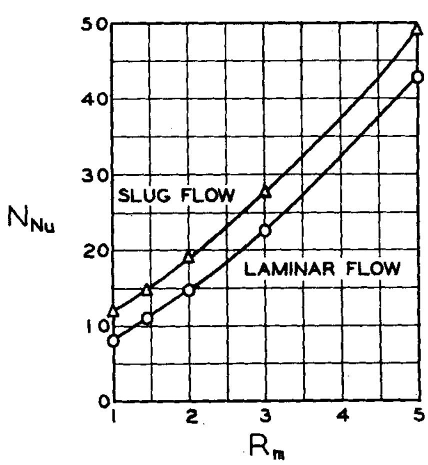


**Fig. 5. Slug and laminar flow Nusselt numbers.**

$f$ = fanning friction factor, dimensionless
$g_c$ = conversion factor, (lb.-mass) (ft.)/(lb.-force) (hr.)$^2$
$h$ = heat transfer coefficient, B.t.u./(hr.) (sq. ft.) (°F.)
$k$ = molecular thermal conductivity, B.t.u./(hr.) (ft.) (°F.)
$k_e$ = eddy thermal conductivity, B.t.u./(hr.) (ft.) (°F.)
$k_{eff}$ = $k + k_e$ = effective thermal conductivity, B.t.u./(hr.) (ft.) (°F.)
$m$ = (cross-sectional flow area)/(wetted perimeter), ft.
$N_{Nu}$ = $(hD_e)/k$ = Nusselt number, dimensionless
$N_{Pe}$ = $(N_{Pr})(N_{Re})$ = Peclet number, dimensionless
$N_{Pr}$ = $(c_p \nu \rho)/k$ = Prandtl number, dimensionless
$N_{Re}$ = $(D_e u_a)/\nu$ = Reynolds number, dimensionless
$p$ = pressure, (lb.-force)/(sq.ft.)
$P$ = pitch for equilateral triangular spacing, ft.
$P'$ = pitch for square spacing, ft.
$q$ = radial heat flow per linear foot at radius $r$, B.t.u./hr.
$r$ = radial distance from tube axis, ft.
$R$ = $r/r_1$ = radius ratio, dimensionless
$t$ = temperature, °F.
$u^+$ = $u/u^*$ = dimensionless velocity, dimensionless

$u^*$ = $\sqrt{(g_c\tau_1)/\rho}$ = friction velocity, ft./hr.
$u$ = velocity, ft./hr.
$V$ = $u/u_a$ = velocity ratio, dimensionless
$x$ = axial distance along tube, ft.
$y$ = $r - r_1$ = distance from wall, ft.
$y^+$ = dimensionless wall distance, dimensionless

**Greek Letters**

$\epsilon_H$ = $k_e/\rho c_p$ = eddy diffusivity of heat, sq.ft./hr.
$\epsilon_M$ = eddy diffusivity of momentum, sq.ft./hr.
$\nu$ = kinematic viscosity, sq.ft./hr.
$\rho$ = density, lb.-mass/cu. ft.
$\tau$ = shear, lb.-force/sq.ft.
$\psi$ = $\epsilon_H/\epsilon_M$, dimensionless

**Subscripts**

$a$ = average
$c$ = transition between laminar and turbulent zones
$L$ = laminar flow
$m$ = on $r$ denotes radius of maximum velocity; on $t$ denotes velocity weighted mean temperature
$q, t$ = designates variables in successive integrations
$S$ = slug flow
$u$ = unadjusted
$1$ = at wall

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# Use of Boundary-Layer Theory to Predict the Effect of Heat Transfer on the Laminar-Flow Field in a Vertical Tube with a Constant-Temperature Wall

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**Variations of density and viscosity with temperature cause distortions in the flow field and affect the rate of heat transfer to fluids in laminar flow in vertical tubes. The magnitude of these distortions is predicted through an approximate solution of the equations of motion and energy.**

Variation of density and viscosity with temperature may cause distortions in the parabolic flow field and may affect the rate of heat transfer and the stability of flow of fluids at low Reynolds numbers in vertical tubes with a constant-temperature wall. An analytical expression for the temperature field and the rate of heat transfer to fluids in laminar flow with constant properties was obtained by Graetz (*2*). Sellars, Tribus, and Klein (*15*) extended the solution of Graetz and provided all the necessary eigenvalues and eigenfunctions to construct the infinite-series solution. Whiteman and Drake (*18*) showed how other velocity profiles affect the rate of heat transfer; Singh (*17*) and Millsaps and Pohlhausen (*9*) included axial conduction and viscous dissipation in the solution, and Schenk and Dumore (*13*) calculated the effect of wall resistance.

Distortions of the parabolic laminar velocity profile due to viscosity variation were discussed by Keevil and McAdams (*6*). Hanratty, Rosen, and Kabel (*5*) and Scheele, Rosen, and Hanratty (*12*) described visual experiments on the effect of heating and cooling water in a vertical tube where density variation (natural convection) was more important than viscosity variation. Under relatively mild conditions of heating and cooling, distortions were found sufficient to change the parabolic velocity profile to such an extent that the flows became turbulent at Reynolds numbers much lower than those for isothermal flow. The results showed two types of instabilities. For cooling in upflow, transition to turbulence is associated with the condition at which the velocity gradient at the wall becomes zero and a reversal of flow occurs. The instability develops rapidly in such cases. For heating in upflow it appears as if a length of pipe is required to allow the instabilities in the flow to grow. Instabilities may be observed downstream of the point where the velocity profile at the center becomes flat

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